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Vladimir Anashin

Causality: The *p*-adic Theory

Springer

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To Alla and Masha, with gratitude for their love and care

Foreword

The book *Causality: The p-adic Theory* by Vladimir Anashin is devoted to the study of the question posed by Albert Einstein: “Does God play dice ?” Or in other words, is there randomness inherent in our world at a fundamental level?

According to modern quantum mechanics, there is an irreducible randomness of the fundamental physical laws. However, Einstein did not agree with this statement and in the discussion with Niels Bohr presented arguments that such a situation is impossible.

Although Einstein failed to convince Bohr that he was right, as a result of this discussion, the famous article of Einstein, Podolsky, and Rozen appeared, which became the initial stage in the development of modern quantum information. These studies were continued by Erwin Schrodinger, John Bell, and many others and now forms the basis of quantum technology.

However, the question of whether Nature is deterministic at a fundamental level or whether it has an element of chance continues to be debated in the literature. In particular, the Nobel laureate Gerald ’t Hooft believes that there is no chance of randomness on the fundamental level. This point of view is shared by the author of this book, Vladimir Anashin, but with a caveat: He presents strong mathematical arguments that if causality is inherent to Nature, then on the base of observational data it is impossible to answer that question “yes” or “no.”

Although the current dominant view is that there is irreducible randomness at the fundamental level, for the future development of the theory of quantum cosmology, black hole physics, chaos, big data theory, and artificial intelligence, it may be useful and even necessary to consider an alternative point of view when at the fundamental level the laws of nature are deterministic, and chaotic and random behavior arises as a consequence of an inevitable error in determining the initial data. The author of the book, Vladimir Anashin, is a professor of computer science at Moscow State University, and is a major specialist in computer science, so based on his expertise in that area he argues that a deterministic point of view is not less rationalized than a non-deterministic one and obtains remarkable results along this path in this book.

The book consists of three parts, it includes the necessary information on p-adic analysis, automata theory, as well as physical applications. The author managed to obtain a beautiful description of causal functions based on the following assumptions. On the one hand, he takes as a basis ’t Hooft’s deterministic approach to the concept of causality based on automata theory. On the other hand, he sets out the basic principles for describing fundamental laws based on postulates such as the observability of only rational numbers, the inability to observe and measure distances smaller than Planck distances, as well as the invariance of the laws of Nature under change of the number field. On this basis the author was able to obtain a complete description of causal functions in terms of special polynomials. There is no doubt that the Anashin’s approach will play an important role in the further development of the theory.

This book may be useful to readers interested in the foundations of quantum theory, computer science, p-adic analysis, and their applications.

Moscow Igor Volovich
June, 2024

Preface

This book is about causality, yet not of causality as a philosophy concept but rather of causality as a mathematical notion. That is, *the main content of the book is mathematical theory of causal functions over discrete time*. The mathematical theory is not a pure mathematical one: It is motivated by applied problems, and a number of mathematical results that are obtained within this theory already have (or may have in future) applications within other mathematical disciplines like automata theory and combinatorics as well as outside mathematics, namely, in applied computer science and maybe (or so I hope) in physics.

By this reason, the book is divided into three parts: The main content of Parts I and II is the said mathematical theory (which basically is the theory of p-adic 1-Lipschitz functions); Part III is devoted to applications of this theory. Though certain mathematical maturity of the readers is mandatory (MS degree in Mathematics, or in Computer Science, or in Physics is desirable), the readers who are interested mostly in applications may use Part II for references only and do not read proofs; however, for these readers it is necessary at least to look through Part I to get acquainted with basics of p-adic theory which is used throughout the book. Chapter 1 which belongs to neither of Parts I–III contains a variety of mathematical notions and facts which are used in the book; this chapter is mostly a collection of reminders to ease referencing. The most important of these reminders, the rigorous definition of causal function over discrete time, is given in this chapter, in Sect. 1.6.2.

Causal function is the main object examined in the book; the latter function may be thought of as a black box which accepts "causes" and produces "effects." The causes/effects are sequences of "elementary causes/effects" which follow one by one, each elementary cause/effect in each instant of time. Therefore, time is assumed to be discrete, and at each discrete instant of time only one of all elementary causes/effects is accepted/produced by the black box. The total number of elementary causes/effects is assumed to be finite; this *finiteness assumption* is assumed since no physical objects the total number of whose is infinite are known: Even the number of all elementary particles in our Universe is finite.

When the black box accepts an elementary cause, the box produces an elementary effect; but *if the next elementary cause is the same as the previous one, the next* *elementary effect may be different from the one which was produced just before*. This means that *when the black box accepts elementary cause, something is changed inside the box*; therefore, the elementary effects that produces the black box depend not only on elementary causes the box accepts but also on this "something" which is constantly changing inside the black box in dependence on elementary causes the black box accepts. These "something(s)" are called "states" of the black box. The *states are epistemic*; i.e., nobody can observe them directly: An observer may only make guesses about the states observing sequences of elementary causes and respective elementary effects. The guesses show that *generally the next state depends not only on the elementary cause the black box accepted but also on the state the black box was just before*, at the previous time instant. For mathematicians and computer scientists it should be clear now that the black box is *sequential machine*, a *letter-to-letter transducer*, a *synchronous automaton*: All these terms mean the same thing that is also known as *Mealy sequential machine* or *Moore sequential machine*. Yes, this is true, but with an adjustment that the total number of states (whatever could these be) of a black box need not be a finite. This does not contradict the finiteness assumptions since the "states" are epistemic, and not ontic, so one may think of infinity and use infinity in calculations despite no infinity exists in physical world: Infinity is a mathematical concept, not a physical entity; but the concept has proven useful over the centuries because it allows us to estimate the values of quantities that depend on variables that are too small or too large compared to others (i.e., when the values of these variables "tend to their limits").

A physicist may think of the black box as a model of open physical system, but with a caution: The physicist should not assume that the whole long strings of elementary causes and elementary effects, i.e., causes and effects as a whole, are completely observable. Very rarely causes and effects are the things that are available to observations with no "distortion." To put it in other words, the observables are just approximations of the causes and effects.

The question that should be asked immediately after the word "approximation" is that: With respect to what metric the approximation is? It is necessary to state rigorously which causes/effects are close to one another and which are far. This metric is defined in the book; actually *three* different metrics are defined.

1. The first one of the three metrics is standard real *Archimedean metric* which can be defined for the causes/effects as follows: Let there are only $p > 1$ elementary causes and elementary effects enumerated by $\{0, 1, \ldots, p-1\}$; then, under the finiteness assumption, the causes/effects may be thought of as strings of some length ℓ. To each string $\alpha_{\ell-1}\alpha_{\ell-2}\ldots\alpha_0$, where indices stand for time instances the elementary cause/effect happens, put into the correspondence the number $0.\alpha_{\ell-1}\alpha_{\ell-2}\ldots\alpha_0 = \alpha_{\ell-1}p^{-1}+\alpha_{\ell-2}p^{-2}+\cdots\alpha_0 p^{-\ell}$ from the unit real segment $[0, 1]$. That is, the most significant digits of the base-p expansion of these numbers correspond to elementary causes/effects which happen later: The more significant is the digit in the base-p expansion of the number which corresponds to cause/effect, the later respective elementary cause/effect happen. The distance between the causes/effects is, by the definition, the ordinary distance between

real numbers $a, b \in [0, 1]$, the real absolute $|a - b|$. That is, the metric characterizes temporal evolution of the system represented by the black box: *Elementary causes/effects that happen later are more significant than the ones that have happened earlier; but the earlier ones may have a crucial influence on future behavior of the system* since each of these early elementary causes/effects change current states of the system and thus may lead to a sort of "dead butterfly effect" from R. Bradbury's science fiction short story *The Sound of Thunder*.

2. The second metric is defined similarly, but the digits of the numbers are taken in the opposite order compared to the first metric: To the sequence $\alpha_{\ell-1}\alpha_{\ell-2}\ldots\alpha_0$ there corresponds the number $0.\alpha_0\alpha_1\ldots\alpha_{\ell-1} = \alpha_0 p^{-1} + \alpha_1 p^{-2} + \cdots \alpha_{\ell-1} p^{-\ell}$, i.e., the earlier the elementary cause/effect happens, the more significant the corresponding digit is. The metric in this case is again the metric on reals; *this metric which reveals the "immediate response" of the system since the later the elementary causes happen, the less significant is the total effect; only the earliest elementary causes matter.*

   Metrics 1–2 have some deficiencies: *The first metric can be well-defined for finite sequences of elementary causes/effects only; therefore, it is not suitable for "predictions" of future behavior of the system or to "reveal the history" of the system.* Though the second metric can be defined for infinite sequences of elementary causes/effects, for a pair of different causes/effects there may correspond one real number, and there are infinitely many such pairs of causes/effects. *The second metric is therefore a pseudo-metric; moreover, it is "incapable to predict the future,"* e.g., to predict results of experiments.

3. To avoid these deficiencies, a yet one more metric is needed further; actually this metric is the main metric that is used throughout the book. The metric is a natural metric on one-side infinite sequences: Given two left-infinite sequences $\mathbf{a} = \cdots\alpha_3\alpha_2\alpha_1\alpha_0$ and $\mathbf{a}' = \cdots\alpha'_3\alpha'_2\alpha'_1\alpha'_0$ over $\{0, 1, \ldots, p-1\}$, put $d_p(\mathbf{a}, \mathbf{a}') = p^{-n}$, where $n = \max\{k \colon \alpha_k = \alpha'_k\}$ if that n exists, otherwise put $d_p(\mathbf{a}, \mathbf{a}') = 0$ if $\alpha_i = \alpha'_i$ for all $k = 0, 1, 2\ldots$. This metric is a metric rather than a pseudo-metric: $d_p(\mathbf{a}, \mathbf{a}') = 0$ if and only if $\mathbf{a} = \mathbf{a}'$. Moreover, the metric d_p is capable "to predict the future and to reveal the past" since to every black box $\mathfrak{B}$ there corresponds a unique well-defined map $f_{\mathfrak{B}}$ of "causes" to "effects" which is continuous with respect to this metric since $d_p(f_{\mathfrak{B}}(\mathbf{a}), f_{\mathfrak{B}}(\mathbf{a}')) \leqslant d_p(\mathbf{a}, \mathbf{a}')$. *The metric d_p is an ultrametric, the non-Archimedean metric* which satisfies strong triangle inequality $d_p(\mathbf{a}, \mathbf{c}) \leqslant \max\{d_p(\mathbf{a}, \mathbf{b}), d_p(\mathbf{b}, \mathbf{c})\}$; therefore, the mapping $f_{\mathfrak{B}}$ is 1-Lipschitz with respect to the metric d_p, i.e., satisfies Lipschitz condition with the constant 1. The most important is that, given any 1-Lipschitz map f of "causes" to "effects" there exists a "black box" $\mathfrak{B}$, the sequential machine, such that $f = f_{\mathfrak{B}}$. *This fact immediately gives a possibility to examine mappings of "causes" to "effects" that the black box (which a model of open system) performs, in order to understand and predict behavior of the system rather than trying unsuccessfully to "open the box in order to look what is inside the box." Facilities for that examination do exist: These are p-adic analysis and p-adic dynamics.*

By this reason, the claim that *causality is essentially non-Archimedean* might be taken as a motto of the book, but not as a motto only: *The non-Archimedean approach to causality results in a sort of a toolbox full of mathematical tools which can be (and some already are) effectively used in concrete applied areas* like the design of pseudo-random number generators (see Chap. 14) for computer simulations, data protection, etc.; experiment design (see Chap. 15); control of automata behavior, design of automata having prescribed properties, smart contracts design (see Chap. 13); straight-line computer programs (SLP) design (see Sect. 9.5). The underlying reason why the "non-Archimedean tools" may work (and already are working) in the said applied areas is that *main computer instructions, both arithmetical (addition and multiplication of integers) and bitwise logical (like bitwise negation* NOT, *bitwise logical conjunction* AND, *bitwise logical disjunction* OR, *etc.) as well as their compositions, are continuous with respect to non-Archimedean 2-adic metric*.

Another aspect of the non-Archimedean view at causality, the dominating view in the book, is related to contemporary *superdeterministic interpretations of quantum mechanics*, see Chap. 16. The reader should be warned that the author of this book is a mathematician rather than a physicist and therefore does not feel himself confident enough to develop a yet one more interpretation of quantum mechanics; but nonetheless the author takes the liberty to use the non-Archimedean approach to some fundamental questions of quantum theory in order to shed some mathematical light onto these questions. These fundamental questions are: Whether Nature on the smallest of the scales is "discrete" or "continuous"? Whether randomness is intrinsic property of Nature at the smallest of scales, or maybe Nature is strictly deterministic? And what is "free will" in case strict determinism rather than randomness is immanent? The answers are given in the form of *interpretations* rather than mathematical theorems; but all the interpretations are based on mathematical assertions which are rigorously proved in the book.

As a whole, these interpretations (supported by mathematical assertions) constitute an evidence for the claim that superdeterministic models of quantum mechanics may have same predictability power as "classical" models. *The very appearance of superdeterministic models of quantum mechanics is in some sense inevitable in view of model-dependent realism concept* (the term was coined by Stephen Hawking and Leonard Mlodinow in their 2010 book, *The Grand Design*). However, this is not the only reason why the author touches the questions related to very foundations of quantum mechanics: *The questions of that kind inevitably arise in connection with concrete applications* like semiconductor devices based on quantum processes that are assumed to be "chaotic." The devices are claimed to produce sequences of "true random" numbers or perform "true random transformations" of bitwords; these devices are aimed to be used in industry as parts of sensitive appliances where the randomness is a must. These practical questions are also mentioned (though briefly) in Chap. 16.

Moscow, Russia Vladimir Anashin
May, 2024

Acknowledgments

Me, the author of this book, is indebted to many people thanks to whom the book appears; and especially for the questions the people asked me. Good questions matter! A good question is always a good starting point of interesting research.

First of all, I am indebted to my late Teacher and Supervisor Professor *Mikhail Mikhailovich Glukhov* not only for teaching and supervising but also for asking me a question which became a starting point of all my future research: How to produce the recurrence sequence of the longest period over a non-Abelian group using only operations of the group? This problem turned out to be fruitful and related to other interesting problems like, e.g., identities in groups, polynomial and affine completeness of groups, etc. Finally, my attempts to solve this problem resulted in ergodic theory for profinite groups with operators [13, 24], and papers on mixed identities in groups [6, 8].

The next good question was due to my colleague and friend *Mikhail Larin* who asked me once: "Why only the groups? What about recurrence sequences of the longest period produced only with the use of operations of other algebraic structures?". To that moment, Mikhail had already proved his criterion of transitivity of polynomials over residue rings modulo N which stayed unpublished until 2002, [132]. I got many ideas from his theorem (and published paper [7] developing these ideas); but the most important for me was the next question by Mikhail: "Do you see that the crucial property you and me are using in our works is that the mapping is compatible with all congruences of algebraic system? What are these mapping then, how to express them in a closed form, for instance, for residue rings?" This question turned out to be very fruitful because when I was trying to find the right answer I understood that the answer should be given in terms of p-adic 1-Lipschitz functions since p-adic integers constitute a profinite ring, the structure which is similar to profinite groups; and to that moment I already had developed the ergodic theory for polynomial transformations of profinite groups with operators. Thus, quite naturally my studies resulted in the ergodic theory of p-adic 1-Lipschitz functions which is the main mathematical theme of the current book. Thank you, Misha, for that question which you asked me in due time!

The third question was asked much later by my colleague and friend *Prof. Andrei Khrennikov*; to that moment our collaboration in algebraic dynamics had already resulted in monograph [24] and a number of papers, see References. Andrei did a lot of research in mathematical physics which was completely out of my area of expertise, not speaking of quantum mechanics; but Andrei's question was about quantum mechanics. Once during my stay in Sweden at Linnaeus University which Andrei was affiliated with, Andrei and me were walking around a beautiful lake and, of course, talking about mathematics. Andrei suddenly asked me: "Why mathematical formalism of quantum mechanics, of essentially nonlinear quantum phenomena, is the theory of *linear* operators? Why the linearity, anyway?". My reaction was unexpected even for me: "Because of causality and finiteness: Finite casual systems can 'compute' nothing more complex than linear functions." "Can you prove this?"—asked me Andrei. "Sure!"—I overconfidently replied. Well, finally I have found the proof that satisfies me [17, 18]. This happened about five years later the moment when the question was asked: That good was the question!

This way, step by step, I start getting involved in mathematical physics, mostly in connection with applications of the p-adic theory to quantum mechanics. Finally I have found myself in a good company which gathered around the Department of Mathematical Physics of Steklov Mathematical Institute, Russian Academy of Sciences. *Corresponding member of the Russian Academy of Sciences I. V. Volovich* who headed the Department at that time organized a number of my talks at the Department and at the Institute introduced me to Academician of the Russian Academy of Sciences V. S. Vladimirov and Dr. E. I. Zelenov, his co-authors of the book *p-adic Analysis and Mathematical Physics* [182], the Bible of the p-adic mathematical physics. During these talks I got a lot of questions; to some of these I gave answers immediately, to some after a while, and I am still thinking over the rest ones hoping to answer them sometimes. But the most important question which inspired me a lot was this one: What are the functions which satisfy Volovich postulates (see Sect. 16.1 about these)? My answer is Theorem 16.7 and, actually, the whole Chap. 16 which incorporates papers [22, 23].

My works related to interplay between p-adic dynamics, computer science, and quantum theory were unexpectedly noticed by people in the applied area of which I never heard before: The semiconductor superlattices, quite complex quantum devices. Once *Prof. Yaohui Zhang* and *Dr. Wei Liu* from the Institute of Nano-Tech and Nano-Bionics Chinese Academy of Sciences found me during one of my stays in Beijing, where I was a visiting professor of the Chinese Academy of Sciences, and asked me a question: "How to *prove* that the superlattice-based device which performs random mappings of bitstrings to bitstrings does perform the mappings which are *truly random*?" Wow, that was a challenge! "To prove" in that context means to convince a customer which is not supposed to believe in this "true randomness" just because a number of physicists assume that the randomness is inherent in Nature at quantum level, or because the devices pass standard statistical tests. Finally, I published work [23] results which are included in Chap. 16. I am not sure whether these results are convincing enough, but my collaboration with Prof. Yaohui Zhang and Dr. Wei Liu still continues. By the way, some results of

the said paper were motivated by a question of *Prof. Franco Vivaldi* from Queen Mary University of London who asked me at some conference what is the entropy of the p-adic 1-Lipschitz maps, and I replied that the entropy is zero since all these maps are conjugate to the odometer, and odometer has zero entropy. But after that I thought that the entropy of the logistic map $L(x) = 4x(1 - x)$ is positive, but the logistic map is a polynomial with integer coefficients, whence it is a p-adic 1-Lipschitz function for every p, hence ... it is conjugate to odometer. This is not a contradiction, of course; but that very question led me to the idea of free choice of metric by an observer with respect to which the observer makes his decision whether the dynamics is chaotic or not, see Sect. 16.4.2 of the current book.

And of course I ought to say thanks to my colleagues from the Institute of Information Engineering Chinese Academy of Sciences, especially to *Prof. Dongdai Lin* for hospitality during all my visiting professorship at the Institute but also for questions; for instance for the question about linear relations among coordinate sequences produced by T-functions, my results from Sect. 14.3 show that there are no such relations in a general T-function, but what about large important subclasses of T-functions? The answer was our joint work [169]: These relations exist if T-function satisfies some (very mild) smoothness conditions. I have not included the result into the current book since it is a somewhat specific one; but the proof of it demonstrates the usefulness of p-adic analysis to handle very specific problems which arise in applied computer science.

Last but not least: I must say lots of thanks to my colleagues from the Faculty of Computational Mathematics and Cybernetics of Lomonosov Moscow State University which I am affiliated with for a long time, for stimulating questions and discussions. My special thanks to *Academician of the Russian Academy of Sciences I. A. Sokolov*, the Dean of our faculty, who constantly supports me in research and teaching, who communicates my papers to "Doklady of the Russian Academy of Sciences," and who asked me questions, too! It was he who initiated the research on digital economy within which I have understood that smart contracts are causal functions whose interactions within digital economy in physical time can be examined by the methods of p-adic theory [20, 21]. Maybe, this examination will lead to some future results of practical value.

To list all the stimulating questions that my friends and colleagues throughout space-time asked me is to write a long paper; therefore, I have to stop and only mention names of some people to whom I am grateful: Dr. rer. nat. Patrick Erik Bradley, Prof. Ivan Chizhov, Prof. Branko Dragovich, Prof. Fabien Durand, Prof. Aihua Fan, Prof. Rostislav Grigorchuk, Prof. Alexander Grusho, Prof. Tor Helleseth, Prof. Sangtae Jeong, Dr. Vladimir Osipov, Prof. Igor Shparlinski, Prof. Livat Tyapaev, Prof. Yuefei Wang, Prof. Jia-Yan Yao, Dr. Igor Yurov, Prof. Michael Zieve, Prof. Wilson Zúñiga-Galindo, and many others to whom I am always thankful but haven't mentioned here, sorry.

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